TWO H-INEQUIVALENT HADAMARD MATRICES OF ORDER 668

ABSTRACT. Four explicit subsets of $\mathbb{Z}_{166}$ form a cyclic difference family and define a bordered Goethals–Seidel Hadamard matrix H of order 668. The four border rows and columns form a Hall set. A paired Hall switch therefore gives another Hadamard matrix $H^\star$. Exact enumeration shows that the Hall row and column sets are unique, and an intrinsic four-row correlation profile distinguishes the two matrices: its maximum is 148 for H and 164 for $H^\star$. Thus H and $H^\star$ are not H-equivalent, even if transposition is admitted.

1. Introduction

A Hadamard matrix of order n is a matrix $M \in \{\pm 1\}^{n\times n}$ satisfying $MM^\mathsf{T} = nI_n$. Two Hadamard matrices are *H-equivalent* if one can be obtained from the other by signed row and column permutations. Transposition is not part of this standard relation.

The four cyclic blocks below define a bordered Goethals–Seidel matrix H of order 668. Its border supplies a Hall set, in the sense of Orrick [4], on which a paired switch produces a second matrix $H^\star$. The point of this note is that the switch changes the equivalence class. A public announcement of the order-668 construction appeared on 12 August 2026 [1].

Theorem 1. *The matrices H and $H^\star$ defined below are Hadamard matrices of order 668 and are not H-equivalent. They remain inequivalent when transposition is admitted as an additional operation.*

All defining data are displayed explicitly. The Hadamard and inequivalence checks use exact integer arithmetic only.

2. The cyclic difference family

Identify $\mathbb{Z}_{166}$ with $\{0, 1, \ldots, 165\}$. Define

$$
\begin{aligned}
X_1 = \{&3, 5, 7, 9, 11, 12, 14, 15, 19, 22, 23, 24, 25, 28, 29, 31, 32, 33, 34, 36, 37, 38,\\
&39, 40, 44, 45, 46, 49, 51, 52, 53, 61, 63, 64, 66, 68, 70, 71, 74, 76, 77, 79, 80, 82,\\
&83, 84, 85, 87, 89, 91, 93, 96, 99, 100, 101, 103, 104, 109, 110, 113, 118, 124, 125,\\
&127, 128, 129, 132, 134, 135, 136, 146, 147, 149, 151, 153, 154, 157, 159, 160, 162, 163, 165\},
\end{aligned}
$$

$$
\begin{aligned}
X_2 = \{&0, 1, 2, 3, 7, 11, 14, 16, 21, 24, 26, 27, 29, 30, 32, 33, 34, 35, 36, 37, 40, 41, 43,\\
&46, 48, 50, 51, 52, 54, 55, 57, 59, 61, 63, 64, 67, 68, 69, 75, 76, 77, 78, 79, 87, 88, 89,\\
&91, 92, 93, 95, 96, 98, 100, 101, 103, 105, 106, 108, 111, 114, 121, 123, 126, 129, 131,\\
&133, 134, 135, 137, 138, 140, 142, 144, 146, 147, 150, 151, 152, 158, 159, 160, 161, 162\},
\end{aligned}
$$

$$
\begin{aligned}
X_3 = \{&0, 1, 2, 3, 5, 7, 10, 11, 12, 14, 17, 18, 19, 20, 22, 23, 24, 30, 31, 33, 34, 35, 37, 38,\\
&41, 49, 51, 52, 54, 55, 56, 58, 59, 60, 61, 64, 68, 69, 70, 71, 74, 75, 76, 80, 81, 82, 87,\\
&89, 91, 92, 96, 98, 99, 104, 108, 109, 110, 111, 112, 115, 119, 123, 132, 134, 135, 137,\\
&138, 139, 141, 142, 143, 144, 147, 151, 152, 153, 154, 157, 158, 159, 163, 164, 165\},
\end{aligned}
$$

$$
\begin{aligned}
X_4 = \{&3, 4, 5, 7, 9, 10, 13, 14, 15, 16, 18, 19, 21, 25, 27, 28, 31, 32, 33, 34, 35, 36, 40,\\
&41, 48, 50, 51, 54, 55, 57, 60, 61, 62, 64, 65, 68, 69, 70, 72, 73, 74, 77, 78, 80, 82, 83,
\end{aligned}
$$

2020 *Mathematics Subject Classification.* 05B20, 05B10.

Key words and phrases. Hadamard matrix, Hall set, switching, difference family, Goethals–Seidel array, equivalence.

$$84, 85, 89, 91, 94, 95, 100, 103, 105, 106, 107, 109, 112, 113, 120, 121, 122, 131, 133,$$
$$134, 137, 138, 140, 144, 145, 147, 148, 151, 152, 153, 155, 156, 157, 160, 161, 163, 165\}.$$

Their cardinalities are $|X_1| = 82$ and $|X_2| = |X_3| = |X_4| = 83$. Direct coefficient counting gives, for every nonzero $t \in \mathbb{Z}_{166}$,

$$\sum_{i=1}^{4} \big|\{x \in X_i : x + t \in X_i\}\big| = 164. \tag{1}$$

Hence the four blocks form a cyclic difference family. Its parameters are

$$(166; 82, 83, 83, 83; 164).$$

For $X \subseteq \mathbb{Z}_{166}$ let C_X be the 166×166 circulant sign matrix whose first row is -1 on X and $+1$ off X. Put

$$A = C_{X_1}, \qquad B = C_{X_2}, \qquad C = C_{X_3}, \qquad D = C_{X_4}.$$

If $N_X(t) = |\{x \in X : x + t \in X\}|$, the inner product of two rows of C_X separated by shift t is $166 - 4|X| + 4N_X(t)$. Thus (1) gives

$$Q := AA^\top + BB^\top + CC^\top + DD^\top = 668 I_{166} - 4J_{166}. \tag{2}$$

The row sums are

$$A\mathbf{1} = 2\mathbf{1}, \qquad B\mathbf{1} = C\mathbf{1} = D\mathbf{1} = 0. \tag{3}$$

3. The Hadamard matrix

Let R be the back-diagonal permutation matrix of order 166. Define the 664×664 Goethals–Seidel core [2]

$$G = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^\top R & -C^\top R \\ -CR & -D^\top R & A & B^\top R \\ -DR & C^\top R & -B^\top R & A \end{pmatrix}. \tag{4}$$

Since circulants commute and $RXR = X^\top$ for every circulant X, direct block multiplication gives

$$GG^\top = I_4 \otimes Q. \tag{5}$$

Let $\mathbf{1} = \mathbf{1}_{166}$ and set

$$K = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

These satisfy

$$KK^\top = TT^\top = SS^\top = 4I_4, \qquad KS^\top = -2T. \tag{6}$$

Define

$$H = \begin{pmatrix} K & T \otimes \mathbf{1}^\top \\ S \otimes \mathbf{1} & G \end{pmatrix}. \tag{7}$$

Lemma 2. *The matrix H in (7) is a Hadamard matrix of order 668.*

Proof. The upper-left Gram block is

$$KK^{\mathsf{T}} + 166TT^{\mathsf{T}} = 4I_4 + 664I_4 = 668I_4.$$

Equations (2) and (5) give the lower-right block

$$(S \otimes \mathbf{1})(S \otimes \mathbf{1})^{\mathsf{T}} + GG^{\mathsf{T}} = 668I_{664}.$$

By (3), $G(q \otimes \mathbf{1}) = 2(q \otimes \mathbf{1})$ for every $q \in \mathbb{R}^4$. The mixed block is consequently $(KS^{\mathsf{T}} + 2T) \otimes \mathbf{1}^{\mathsf{T}} = 0$. Hence $HH^{\mathsf{T}} = 668I_{668}$. □

4. The Hall switch

Number the rows and columns from 1 to 668 and let

$$P = \{1, 2, 3, 4\}.$$

For $1 \leq f \leq 4$, let C_f be the fth consecutive core column field of size 166, and let R_f be the corresponding core row group.

The entrywise product of the four rows in P is $+1$ on the four border columns and -1 on the 664 core columns: the columns of K have product $+1$ and those of T have product -1. Dually, the product of the first four columns is $+1$ on P and -1 below it, using the row products of K and S. Thus the border rows and columns form a Hall set.

Define $H^\star = (h^\star_{rc})$ by the paired switch

$$h^\star_{rc} = \begin{cases} -h_{rc}, & (r, c) \in (P \times C_1) \cup (R_1 \times P), \\ h_{rc}, & \text{otherwise.} \end{cases} \tag{8}$$

Exactly $2 \cdot 4 \cdot 166 = 1328$ entries change. This is Orrick's Hall switch, which preserves the Hadamard property [4, Proposition 3.4]. For completeness, it also follows directly from the block form. Inner products within the same switched state are unchanged. Mixed core inner products use $S_1 \cdot S_g = 0$; a border/core inner product changes by

$$-2\big(K_i \cdot S_1 + 2T_{i1}\big) = 0$$

by (6). Therefore

$$H^\star (H^\star)^{\mathsf{T}} = 668I_{668}. \tag{9}$$

5. Inequivalence

Call four rows a *type-1 quadruple* if their entrywise product has four entries of one sign and 664 of the other. The four minority coordinates are its exceptional set.

Lemma 3 (Exact computation)**.** *In each of H and $H^\star$ there is exactly one type-1 row quadruple and exactly one type-1 column quadruple. In all four cases it is P, with exceptional set P.*

The uniqueness makes the Hall/core partition intrinsic. For either matrix $M = (m_{rc})$, define

$$\Phi_M(t) = \#\left\{ (a, \{i, j, k\}) : \begin{array}{c} a \in P, \quad 5 \leq i < j < k \leq 668, \\ \left|\sum_{c=1}^{668} m_{ac} m_{ic} m_{jc} m_{kc}\right| = t \end{array} \right\}. \tag{10}$$

Column signs cancel in every four-row product, row signs change only its overall sign, and row and column permutations merely reorder the terms. Thus Φ_M is an H-equivalence invariant.

There are $4\binom{664}{3} = 194{,}289{,}056$ terms in each profile. The exact upper tails are

t	$\Phi_H(t)$	$\Phi_{H^\star}(t)$
132	46	76
140	10	8
148	2	0
156	0	2
164	0	2

Both profiles vanish above the last displayed nonzero entry. Hence

$$\max\operatorname{supp}(\Phi_H) = 148, \qquad \max\operatorname{supp}(\Phi_{H^\star}) = 164, \tag{11}$$

so H and $H^\star$ are not H-equivalent.

If transposition is admitted, it remains to compare H with $(H^\star)^\mathsf{T}$. The exact profiles give

$$\Phi_H(4) = 47{,}235{,}186, \qquad \Phi_{(H^\star)^\mathsf{T}}(4) = 47{,}224{,}508. \tag{12}$$

This proves Theorem 1.

The four paired Hall switches can be applied independently. All 16 switch masks are Hadamard. A pinned colored-graph canonicalization [3] divides them into exactly two H-classes:

$$\text{even masks: } 0, 3, 5, 6, 9, 10, 12, 15,$$
$$\text{odd masks: } 1, 2, 4, 7, 8, 11, 13, 14.$$

6. Finite verification

All computations use integer bitsets; no floating-point or probabilistic test is involved. First, (1) is checked at all 165 nonzero shifts. The expanded matrices are then checked over all $\binom{668}{2} = 222{,}778$ row pairs and all 222,778 column pairs; every Hamming distance is 334 before and after switching.

For Lemma 3, encode each row by the bitset of its negative entries. A quadruple is type 1 exactly when the XOR of its four bitsets has weight 4 or 664. All row-pair products are bucketed on five disjoint coordinate chunks, and every candidate is rechecked on all 668 coordinates. The profile (10) is evaluated directly using

$$|668 - 2\ \mathrm{wt}(b_a \mathbin{\mathsf{xor}} b_i \mathbin{\mathsf{xor}} b_j \mathbin{\mathsf{xor}} b_k)|\,,$$

where b_r is the negative-entry bitset of row r.

With row-major bytes encoding $+1$ by 1 and -1 by 0, the matrix hashes are

H	`19f435b31eb6561dd97356b761e9b0f174824e42c215c33fbafc20f9b1b20744`
$H^\star$	`08428eb6779986c50ee2997584daa5c87e00520cc1e6f62d7793dc519f85ea45`

References

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